SOLUTIONS & MARKING KEY

Calculator Free Section

1. **[3 marks]**

Use an inverse matrix method to solve the matrix equation

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 18 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} m \\ n \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 18 \\ 2 \end{bmatrix}$$

$$= -\frac{1}{10}\begin{bmatrix} 1 & -4 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 18 \\ 2 \end{bmatrix}$$

$$= -\frac{1}{10}\begin{bmatrix} 10 \\ -50 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

$\therefore$ $m = -1$, $n = 5$

✓ premultiplies by inverse of 2x2 matrix.

✓ Correct matrix result.

✓ Correct solution stated.

2. **[7 marks]**

Determine:

(a) $\text{Im}\left[cis\left(\frac{5\pi}{6}\right)\right]$ $= Im\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$

$= \sin\left(\frac{5\pi}{6}\right)$

$= \boxed{\frac{1}{2}}$

✓ simplifies to $\sin\frac{5\pi}{6}$

✓ correct answer

[2]

(b) $Arg\left[4cis\left(\frac{-2\pi}{3}\right)\right]^5$ $= Arg\left[4^5 cis\left(\frac{-10\pi}{3}\right)\right]$

$= \frac{-10\pi}{3}$

$= \boxed{\frac{2\pi}{3}}$

✓ uses de Moivre's Theorem to work out $\left(-\frac{2\pi}{3}\right)^5$.

✓ correct answer.

[2]

(c) Describe the locus of points z on the Argand plane defined by the rule:

$$\left(z+\bar{z}\right)^2 - \left(z-\bar{z}\right)^2 = 16$$

If $z = x+iy$

then $z+\bar{z} = 2x$

and $z-\bar{z} = 2iy$

$\therefore$ $(z+\bar{z})^2 - (z-\bar{z})^2 = 16$

$(2x)^2 - (2yi)^2 = 16$

$4x^2 + 4y^2 = 16$

$x^2 + y^2 = 4$

$x^2 + y^2 = 2^2$

$\Rightarrow$ circle radius 2 and centre $(0,0)$

✓ simplifies $z+\bar{z}$ and $z-\bar{z}$

✓ arrives at cartesian equation.

✓ describes the locus of points.

[3]

3. **[6 marks]**

For each of the following functions, find $\frac{dy}{dx}$ in terms of x.

(a) $y = 2\cos\left(e^{x^2}\right)$

✓ Correct use of Chain Rule
✓ Correct use of Product Rule

$$\frac{dy}{dx} = -2\sin\left(e^{x^2}\right) \cdot \left(e^{x^2}\right) \cdot 2x$$

$$= \underline{\underline{-4x\,e^{x^2} \cdot \sin\left(e^{x^2}\right)}}$$

[2]

(b) $(3x^2 + 4)\ln y = 5x$

$$\ln y = \frac{5x}{3x^2+4}$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{(3x^2+4)(5) - (5x)(6x)}{(3x^2+4)^2}$$

$$\therefore \frac{dy}{dx} = \frac{15x^2 + 20 - 30x^2}{(3x^2+4)^2} \cdot y$$

$$= \underline{\underline{\frac{-15x^2+20}{(3x^2+4)^2} \cdot e^{\left(\frac{5x}{3x^2+4}\right)}}}$$

✓ writes $\ln y$ in terms of x.
✓✓ Correctly applies logarithmic differentiation.
✓ Expresses answer in terms of x.

[4]

4. **[7 marks]**

(a) Write $\frac{1 + i\sqrt{3}}{1 + i}$ in cis form.

✓ Correctly converts both numerator and denominator to polar form.
✓ Expresses answer in correct polar notation.

$$\frac{1+i\sqrt{3}}{1+i} = \frac{2\,\text{cis}\left(\frac{\pi}{3}\right)}{\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)} = \underline{\underline{\sqrt{2}\,\text{cis}\left(\frac{\pi}{12}\right)}}$$

[2]

(b) Hence, determine the exact value of $\cos\left(\frac{\pi}{12}\right)$.

✓ 'Realises' the denominator.
✓ Understands that $Re\left(\frac{1+i\sqrt{3}}{1+i}\right)$ is equal to $\sqrt{2}\cos\left(\frac{\pi}{12}\right)$.
✓ correct exact value.

$$\frac{1+i\sqrt{3}}{1+i} \times \frac{1-i}{1-i} = \frac{1+\sqrt{3}}{2} + \left(\frac{\sqrt{3}-1}{2}\right)i$$

$$\therefore \sqrt{2}\cos\left(\frac{\pi}{12}\right) = Re\left(\frac{1+i\sqrt{3}}{1+i}\right)$$

$$= \frac{1+\sqrt{3}}{2}$$

$$\Rightarrow \cos\frac{\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}} = \boxed{\frac{\sqrt{2}\left(1+\sqrt{3}\right)}{4}}$$

[3]

(c) By using the result from (a), or otherwise, calculate $\left(\frac{1 + i\sqrt{3}}{1 + i}\right)^{12}$.

✓ Uses de Moivre's Theorem
✓ Simplifies answer

$$\left(\frac{1+i\sqrt{3}}{1+i}\right)^{12} = \left(\sqrt{2}\,\text{cis}\,\frac{\pi}{12}\right)^{12}$$

$$= 64\,\text{cis}\,\pi$$

$$= \underline{\underline{-64}}$$

[2]

5. **[5 marks]**

(a) Prove that $\tan^2\theta = \dfrac{1-\cos 2\theta}{1+\cos 2\theta}$

✓ Correct substitution for $\cos 2\theta$ used.

✓ Clear method of proof.

RHS: $\dfrac{1-\cos 2\theta}{1+\cos 2\theta} = \dfrac{1-(1-2\sin^2\theta)}{1+(2\cos^2\theta-1)}$

$= \dfrac{2\sin^2\theta}{2\cos^2\theta}$

$= \tan^2\theta = LHS$

※ Q.E.D.

[2]

(b) Hence determine the exact value of $\tan\left(\dfrac{\pi}{8}\right)$

✓ uses $\frac{\pi}{4}$ as 2θ.

✓ exact value of $\cos\frac{\pi}{4}$ used.

✓ rationalised denominator to arrive at final answer.

$\tan^2\left(\frac{\pi}{8}\right) = \dfrac{1-\cos\left(\frac{\pi}{4}\right)}{1+\cos\left(\frac{\pi}{4}\right)}$

$= \dfrac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}$

$= \dfrac{\sqrt{2}-1}{\sqrt{2}+1} \times \dfrac{\sqrt{2}-1}{\sqrt{2}-1}$

$= 3-2\sqrt{2}$

$\therefore \tan\left(\frac{\pi}{8}\right) = \underline{\underline{\sqrt{3-2\sqrt{2}}}}$

[3]

6. **[6 marks]**

Evaluate the following limits, showing full reasoning.

(a) $\lim_{x\to 0}\left(\dfrac{\sin|x|}{x}\right)$

✓ calculates limit from LH and RH.

✓ Shows that LH and RH. limits are not equal.

✓ Correct answer.

$\lim_{x\to 0^-}\dfrac{\sin|x|}{x} = \lim_{x\to 0^-}\dfrac{-\sin x}{x} = -1$

$\lim_{x\to 0^+}\dfrac{\sin|x|}{x} = \lim_{x\to 0^+}\dfrac{\sin x}{x} = 1$

since $\lim_{x\to 0^-}\dfrac{\sin|x|}{x} \neq \lim_{x\to 0^+}\dfrac{\sin|x|}{x}$

$\therefore \lim_{x\to 0}\dfrac{\sin|x|}{x}$ does not exist.

[3]

(b) $\lim_{\theta\to 0}\dfrac{\tan(3\theta)}{\tan(5\theta)} = \lim_{\theta\to 0}\dfrac{\sin 3\theta}{\cos 3\theta}\cdot\dfrac{\cos 5\theta}{\sin 5\theta}$

$= \lim_{\theta\to 0}\dfrac{\sin 3\theta}{3\theta}\cdot\dfrac{3\theta}{\cos 3\theta}\cdot\dfrac{\cos 5\theta}{5\theta}\cdot\dfrac{5\theta}{\sin 5\theta}$

$= \lim_{\theta\to 0}\dfrac{\sin 3\theta}{3\theta}\cdot\dfrac{5\theta}{\sin 5\theta}\cdot\dfrac{3\cos 5\theta}{5\cos 3\theta}$

$= 1\cdot 1\cdot\dfrac{3}{5}$

$= \boxed{\dfrac{3}{5}}$

✓ Re-writes $\tan 3\theta$ and $\tan 5\theta$ in terms of $\sin 3\theta, \cos 3\theta$ and $\sin 5\theta, \cos 5\theta$ respectively.

✓ uses $\lim_{x\to 0}\frac{\sin nx}{n} = 1$

✓ Correct final answer.

[3]

7. **[6 marks]**

Relative to an origin O, point A has cartesian coordinates $(1, 2, 2)$ and point B has cartesian coordinates $(-1, 3, 4)$.

(a) Find an expression for the vector $\overrightarrow{AB}$.

$$\overrightarrow{AB} = \begin{pmatrix}-1\\3\\4\end{pmatrix} - \begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$= \begin{pmatrix}-2\\1\\2\end{pmatrix}$$

✓ correct vector $\overrightarrow{AB}$ provided.

[1]

(b) Show that the cosine of the angle between the vectors $\overrightarrow{OA}$ and $\overrightarrow{AB}$ is $\frac{4}{9}$.

$$\overrightarrow{OA} \cdot \overrightarrow{AB} = \begin{pmatrix}1\\2\\2\end{pmatrix} \cdot \begin{pmatrix}-2\\1\\2\end{pmatrix} = 4$$

$$\overrightarrow{OA} \cdot \overrightarrow{AB} = (\sqrt{9})(\sqrt{9}) \cos\theta$$

$$\Rightarrow \quad 9 \cos\theta = 4$$

$$\cos\theta = \frac{4}{9}$$

※ Q.E.D.

✓ uses dot products correctly

✓ correct method to 'show' $\cos\theta = \frac{4}{9}$

[2]

(c) Hence determine the exact area of the triangle OAB.

since $\cos\theta = \frac{4}{9}$,

$$\sin\theta = \sqrt{1 - \left(\frac{4}{9}\right)^2}$$

$$= \frac{\sqrt{65}}{9}$$

$$\therefore \text{ Area of } \Delta OAB = \frac{1}{2} \cdot 3 \cdot 3 \cdot \frac{\sqrt{65}}{9}$$

$$= \boxed{\frac{\sqrt{65}}{2}}$$

✓ uses Area $= \frac{1}{2} AB \sin C$ formula

✓ calculates $\sin\theta$

✓ calculates Area of ΔOAB

[3]

8. **[5 marks]**

Consider the triangle OAB with $\overrightarrow{OA} = 3i + 2j + \sqrt{3}\,k$ and $\overrightarrow{OB} = \alpha\, i$ where α, which is greater than zero, is chosen so the triangle OAB is isosceles, with $\left|\overrightarrow{OB}\right| = \left|\overrightarrow{OA}\right|$

(a) Show that $\alpha = 4$.

$$\left|\overrightarrow{OB}\right| = \left|\overrightarrow{OA}\right| = \sqrt{16} = 4$$

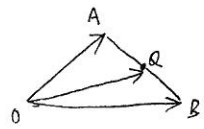
$$\text{and } \left|\overrightarrow{OB}\right| = \alpha$$

$$\Rightarrow \quad \alpha = 4$$

✓ calculates $\left|\overrightarrow{OA}\right|$ and concluded $\alpha = \left|\overrightarrow{OA}\right|$.

[1]

(b) (i) Find $\overrightarrow{OQ}$, where Q is the midpoint of the line segment AB.



$$\overrightarrow{OQ} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$$

$$= \frac{1}{2}\begin{pmatrix}7\\2\\\sqrt{3}\end{pmatrix} = \begin{pmatrix}3.5\\1\\\sqrt{3}/2\end{pmatrix}$$

✓ correctly determines $\overrightarrow{OQ}$.

[1]

(ii) Show that $\overrightarrow{OQ}$ is perpendicular to $\overrightarrow{AB}$.

$$\overrightarrow{OQ} \cdot \overrightarrow{AB} = \begin{pmatrix}3.5\\1\\\sqrt{3}/2\end{pmatrix} \cdot \begin{pmatrix}1\\-2\\-\sqrt{3}\end{pmatrix}$$

$$= 3.5 - 2 - \frac{3}{2}$$

$$= 0$$

$$\therefore \quad \overrightarrow{OQ} \perp \overrightarrow{AB}$$

※

✓ uses dot product.

✓ shows dot product $= 0$.

✓ correct conclusion.

[3]

9. **[7 marks]**

A large sporting goods manufacturer specialising in the sale and supply of hockey sticks promotes three major brands: the Harvey, the Aaron and the George. The number of sales varies according to the seasons.

In winter, 90 Harvey, 40 Aaron and 70 George sticks were sold.
In spring, the numbers were respectively 100, 80 and 110.
In summer, the sales were 30, 60 and 120 respectively.

(a) Display this information in a suitable matrix.

✓ correctly labelled matrix.

$$S = \begin{array}{c} \begin{array}{ccc} H & A & G \end{array} \\ \begin{bmatrix} 90 & 40 & 70 \\ 100 & 80 & 110 \\ 30 & 60 & 120 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Winter} \\ \text{Spring} \\ \text{Summer} \end{array}$$

[1]

(b) If the takings in winter, spring and summer were \$25760, \$37910 and \$28770 respectively, use a matrix method to calculate the cost of each brand of hockey stick.

let h, a and g be the cost of each type of stick respectively.

$$S \begin{bmatrix} h \\ a \\ g \end{bmatrix} = \begin{bmatrix} 25,760 \\ 37,910 \\ 28,770 \end{bmatrix}$$

✓ sets up correct matrix equation.

$$\therefore \begin{bmatrix} h \\ a \\ g \end{bmatrix} = S^{-1} \begin{bmatrix} 25,760 \\ 37,910 \\ 28,770 \end{bmatrix} = \begin{bmatrix} 115 \\ 128 \\ 147 \end{bmatrix}$$

✓ pre-multiplies by the inverse of matrix from (a)

✓ Correct statement of solution.

$\therefore$ Harvey is \$115, Aaron is \$128 and George is \$147

[3]

(c) The number of hockey sticks sold is expected to increase by 10% in the following year. The manufacturer also decided to increase the cost of each brand of hockey stick. If the new costs of the Harvey, Aaron and George are \$130, \$150 and \$175 respectively, carry out a suitable matrix operation to calculate the expected revenue for the following year.

$$1 \cdot 1 \times S \times \begin{bmatrix} 130 \\ 150 \\ 175 \end{bmatrix} = \begin{bmatrix} 32,945 \\ 48,675 \\ 37,290 \end{bmatrix}$$

✓ multiplies matrix from (a) by 1.1

✓ multiplies by new cost matrix.

and
$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 32,945 \\ 48,675 \\ 37,290 \end{bmatrix} = \begin{bmatrix} 118\,910 \end{bmatrix}$$

✓ totals individual elements to arrive at revenue.

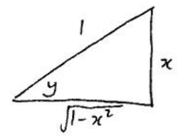
$\therefore$ Expected Revenue is \$118,910

[3]

10. **[7 marks]**

(a) Determine $\dfrac{dy}{dx}$ in terms of x if $y = \sin^{-1} x$ for $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$.

$$x = \sin y$$
$$1 = \cos y \left(\frac{dy}{dx}\right)$$
$$\frac{dy}{dx} = \frac{1}{\cos y}$$



but $\cos y = \sqrt{1-x^2}$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

✓ re-writes equation in terms of y

✓ implicitly differentiates equation with respect to x.

✓ writes cos y in terms of x.

✓ Final answer in terms of x.

[4]

(b) Determine the following limit, given that a and h are constants.

$$\lim_{x \to 0} \frac{2\cos(a+h) - 1 - 2\cos a + 1}{h}$$

$$= \left[\frac{d}{dx}\left(2\cos x - 1\right)\right]_{x=a}$$

$$= \left[-2\sin x\right]_{x=a}$$

$$= -2\sin a$$

✓ concludes that it is the derivative of $2\cos x - 1$

✓ correctly differentiates $2\cos x - 1$

✓ Final answer in terms of a

[3]

11. [12 marks]

Let $A = \begin{bmatrix} m & -3 \\ 4 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

(a) Evaluate each of the following where possible. If not possible, state this clearly and indicate the reason for your decision.

(i) $2A - D$

$= \begin{bmatrix} 2m & -6 \\ 8 & 14 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2m-1 & -9 \\ 8 & 13 \end{bmatrix}$

✓ performs correct operations and simplifies answer.

[1]

(ii) BD

Not possible, since Number of columns in B ≠ No. of rows in D.

✓ states 'not possible' and provides correct reason.

[1]

(b) What value(s) of m make the matrix A singular?

$\det(A) = 0$

$7m + 12 = 0$

$7m = -12$

$m = \frac{-12}{7}$

✓ uses determinant of matrix A = 0

✓ solves for m

[2]

(c) If matrix E is $\begin{bmatrix} 2 & 3 \\ 4 & -7 \end{bmatrix}$ and $AC = E$, then determine the value of m.

multiplying first row of A with first column of C,

$m = 2$

✓ multiplies matrix C by matrix A (OR Row 1 of A x Column 1 of C).

✓ deduces that $m = 2$.

[2]

(d) D represents a transformation matrix. Describe the transformation represented by matrix D.

Horizontal shear (parallel to x-axis) with factor 3.

✓ complete description of transformation

[1]

(e) If an object is transformed by matrix D followed by matrix F, this would have the same effect as if it were transformed by the matrix $\begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$. Describe the effect of transformation matrix F.

$FD = \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$

$\therefore F = \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

F is a dilation of scale factor 2

✓ uses $FD = \begin{bmatrix} 2 & 6 \\ 0 & 2 \end{bmatrix}$

✓ uses inverse matrix method to solve for F

✓ Describes effect of F.

[3]

(f) Describe the transformations which would 'undo' the effect of transformation matrix C followed by transformation matrix D.

$(DC)^{-1} = \left(\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)^{-1}$

$= \begin{bmatrix} 1 & -3 \\ 0 & -1 \end{bmatrix}$

∴ Shear horizontally factor −3.

then reflect vertically about the x-axis ($y = 0$).

✓ determines $(DC)^{-1}$

✓ Correct descriptions of effect of $(DC)^{-1}$, listed in the correct order.

[2]

12. **[5 marks]**

Use the method of proof by contradiction to show that the sum of an irrational number and a rational number is irrational.

let the number R be rational and the number I be irrational.

Then $R = \frac{a}{b}$ (where a and b are integers without common factors)

Assume : ① There is no pair of values c and d such that $I = \frac{c}{d}$.

② That $R + I$ is rational.

Then: $R + I = \frac{e}{f}$ (e, f are integers without common factors)

$$\Rightarrow \quad I = (R+I) - I$$

$$= \frac{e}{f} - \frac{a}{b}$$

$$= \frac{eb - af}{bf}$$

If we let $eb - af = c$ and $bf = d$,

then $I = \frac{c}{d}$, which is a contradiction.

$\therefore$ The sum of a rational and irrational is irrational

\# Q.E.D.

✓ Set up "R" and "I" as rational & irrational, respectively

✓ Assume $R+I$ is rational, i.e., $R+I = \frac{e}{f}$

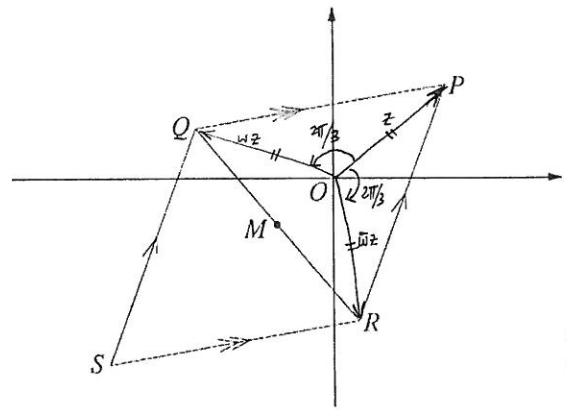
✓✓ Algebraic manipulation to show that $R+I$ = an irrational number

✓ State the contradiction clearly to conclude proof.

13. **[6 marks]**

The point P on the Argand diagram represents the complex number z. The points Q and R represent points wz and $\bar{w}z$ respectively, where $w = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$.

The point M is the midpoint of QR.



✓ use $\vec{OM} = \vec{OQ} + \vec{QM}$

✓ simplify $\vec{OM}$ in terms of w, $\bar{w}$ and z

✓ simplify $\vec{OM}$ in terms of z.

(a) Find the complex number representing M in terms of z.

$w = \text{cis}\left(\frac{2\pi}{3}\right)$

$$\vec{OM} = \vec{OQ} + \vec{QM}$$

$$= \vec{OQ} + \tfrac{1}{2}\vec{QR}$$

$$= wz + \tfrac{1}{2}(\bar{w}z - wz)$$

$$= \tfrac{1}{2}(\bar{w}z + wz)$$

Now : $\frac{1}{2}(\bar{w}z + wz)$

$$= \tfrac{1}{2}z \cdot 2\,\text{Re}(w)$$

$$= \tfrac{1}{2}z \cdot (-1)$$

$$= \underline{\underline{-\tfrac{1}{2}z}}$$

[3]

(b) The point S is chosen so that $PQSR$ is a parallelogram. Find the complex number represented by S, in terms of z.

$$\vec{PM} = \vec{MS}$$

$$\therefore \vec{OS} = \vec{OM} + \vec{MS}$$

$$= -\tfrac{1}{2}z + \left(-\tfrac{3}{2}z\right)$$

$$= \underline{\underline{-2z}}$$

✓ uses the fact that M is the midpoint of $\vec{PS}$.

✓ uses $\vec{OS} = \vec{OM} + \vec{MS}$ (or other equivalent relationship)

✓ simplifies answer, leaving it in terms of z.

[3]

14. **[6 marks]**

Determine the equation of the tangent to the curve defined by $2x^2 + \sqrt{2xy} = 36$ at the point P whose coordinates are (4, 2).

Differentiate Implicitly :

$$4x + \frac{1}{2}(2xy)^{-1/2}\left[2x\left(\frac{dy}{dx}\right) + 2y\right] = 0$$

$$4x + \frac{1}{\sqrt{2xy}}\left[x\left(\frac{dy}{dx}\right) + y\right] = 0$$

$$x\left(\frac{dy}{dx}\right) + y = -4x\sqrt{2xy}$$

$$x\left(\frac{dy}{dx}\right) = -4x\sqrt{2xy} - y$$

$$\therefore \frac{dy}{dx} = -4\sqrt{2xy} - \frac{y}{x}$$

At $(4, 2)$, $\frac{dy}{dx} = -16.5$

$$\therefore (y-2) = -16.5(x-4)$$

$$2y - 4 = -33x + 132$$

$\underline{\underline{33x + 2y = 136}}$ (OR) $y = -\frac{33}{2}x + 68$

✓✓ Differentiates implicitly

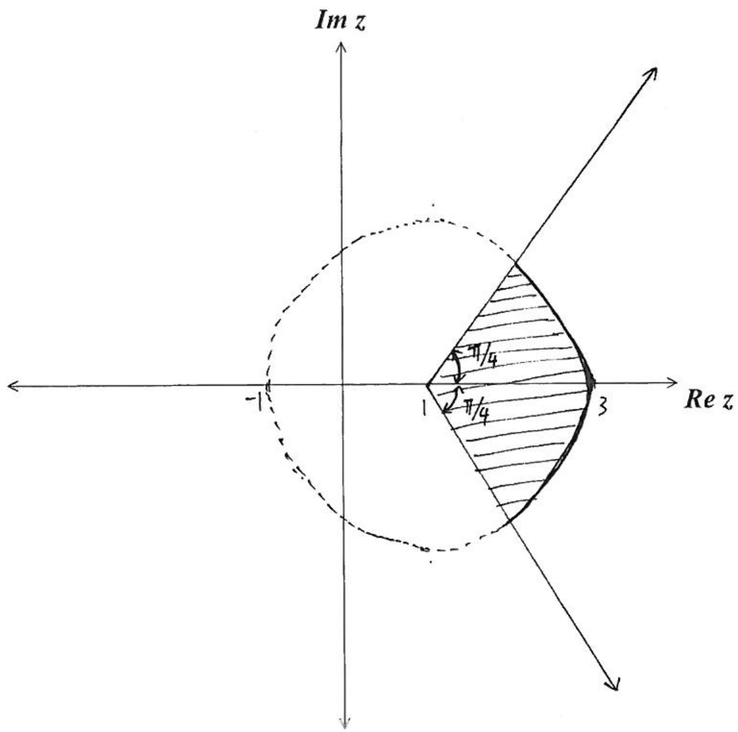
✓✓ Solves for $\frac{dy}{dx}$ in terms of x, y

✓ substitutes $(4,2)$ into $\frac{dy}{dx}$ expression

✓ simplifies equation of line, presenting it in an appropriate format.

15. **[3 marks]**

Sketch the region in the complex plane provided where the inequalities $|z-1| \le 2$ and $-\frac{\pi}{4} \le Arg|z-1| \le \frac{\pi}{4}$ hold simultaneously.



✓ correctly draws rays at $\theta = \frac{\pi}{4}$ and $\theta = -\frac{\pi}{4}$, and commences rays at $Re(z) = 1$

✓ correctly draws circle of radius 2, centre (1,0)

✓ Shades the correct intersection of areas.

16. **[10 marks]**

A defensive missile battery launches a ground-to-air missile A to intercept an incoming enemy missile B. At the moment of A's launch, the position vectors of A and B (in metres), relative to the defensive command headquarters, are $\begin{pmatrix}600\\0\\0\end{pmatrix}$ and $\begin{pmatrix}2200\\4000\\600\end{pmatrix}$ respectively. The velocities (in metres per second) maintained by both A and B are $\begin{pmatrix}-196\\213\\18\end{pmatrix}$ and $\begin{pmatrix}-240\\100\\0\end{pmatrix}$.

(a) Show that the ground-to-air missile does not intercept the enemy missile, and calculate 'how much it misses by'.

$\vec{AB} = \begin{pmatrix}1600\\4000\\600\end{pmatrix}$ and ${}_A\underset{\sim}{v}_B = \begin{pmatrix}44\\113\\18\end{pmatrix}$

For interception, $\vec{AB} = t\,{}_A\underset{\sim}{v}_B$

$\begin{pmatrix}1600\\4000\\600\end{pmatrix} = t\begin{pmatrix}44\\113\\18\end{pmatrix}$

✓ uses relative velocities to set up an equation in t.
✓ solves for t be equating components.
✓ deduces that interception will not occur.

equating i components, $t = \frac{400}{11} = 36.\overline{36}$ sec

equating k components, $t = \frac{100}{3} = 33.\dot{3}$ sec

since t is not unique, the missiles will <u>NOT</u> intercept. ※ Q.E.D.

If P is the point of closest approach,

$\vec{BP} = -\begin{pmatrix}1600\\4000\\600\end{pmatrix} + t\begin{pmatrix}44\\113\\18\end{pmatrix}$

$\vec{BP} \cdot {}_A\underset{\sim}{v}_B = 0$

$\begin{pmatrix}-1600+44t\\-4000+113t\\-600+18t\end{pmatrix} \cdot \begin{pmatrix}44\\113\\18\end{pmatrix} = 0$

✓ uses relative velocities to set up an equation in t.
✓ determines t at point of closest approach.
✓ determines the closest distance.

$\Rightarrow t = 35.478$ sec

$\therefore |\vec{BP}| = \underline{\underline{55.59\text{m}}}$ (2dp)

[6]

(b) Suppose instead that the computer on missile A detects that it is off target and, 20 seconds into the flight, A changes its velocity and intercepts B after a further 15 seconds. Determine the constant velocity that A must maintain during this final 15 seconds for the interception to occur.

After 20 seconds:

$\underset{\sim}{r}_A = \begin{pmatrix}-3320\\4260\\360\end{pmatrix}$ and $\underset{\sim}{r}_B = \begin{pmatrix}-2600\\6000\\600\end{pmatrix}$

$\therefore \vec{AB} = \begin{pmatrix}720\\1740\\240\end{pmatrix}$

In order to intercept 15 seconds later:

$\vec{AB} = 15\,{}_A\underset{\sim}{v}_B$

$\therefore {}_A\underset{\sim}{v}_B = \frac{1}{15}\begin{pmatrix}720\\1740\\240\end{pmatrix}$

$= \begin{pmatrix}48\\116\\16\end{pmatrix}$

$\Rightarrow \underset{\sim}{v}_A = \begin{pmatrix}48\\116\\16\end{pmatrix} + \begin{pmatrix}-240\\100\\0\end{pmatrix}$

$\underset{\sim}{v}_A = \underline{\underline{\begin{pmatrix}-192\\216\\16\end{pmatrix} \text{ m/s}}}$

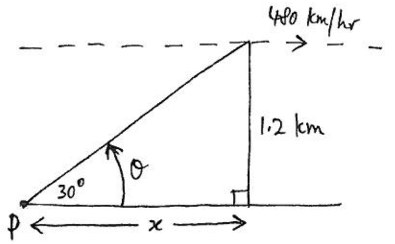
✓ determines relative displacement between A and B at 20 seconds.
✓✓ sets up an equation using relative displacement and relative velocity, and solves this equation for ${}_A\underset{\sim}{v}_B$.
✓ uses vector addition to arrive at velocity of A.

[4]

17. **[6 marks]**

An aircraft is flying horizontally at a constant height of 1200 metres above a fixed observation point P. At a certain instant, the angle of elevation θ is 30° and decreasing, and the speed of the aircraft is 480 km/hr.

(a) Draw a diagram to illustrate this information.



✓ diagram showing $\theta = 30^\circ$, height of aircraft = 1200m.

[1]

(b) How fast is θ decreasing at this instant, in radians per second.

$$\tan\theta = \frac{1200}{x}$$

$$x \tan\theta = 1200$$

$$x \cdot \sec^2\theta \cdot \left(\frac{d\theta}{dt}\right) + \tan\theta \cdot \left(\frac{dx}{dt}\right) = 0$$

Now! at $\theta = \frac{\pi}{6}$, $x = 1200\sqrt{3}$, and $\sec^2\theta = \frac{4}{3}$

So: $\left(1200\sqrt{3}\right)\left(\frac{4}{3}\right)\left(\frac{d\theta}{dt}\right) + \left(\frac{1}{\sqrt{3}}\right)\left(\frac{400}{3}\right) = 0$

$$\therefore \frac{d\theta}{dt} = \frac{-400}{3\sqrt{3}} \cdot \frac{3}{4} \cdot \frac{1}{1200\sqrt{3}}$$

$$= \frac{-1}{36} \text{ rad/sec}$$

✓ sets up a relationship between "x" and $\tan\theta$.
✓ differentiates implicitly.
✓ determines value of "x" and $\sec^2\theta$ when $\theta = 30^\circ$.
✓ substitutes "$\frac{dx}{dt}$" for $\frac{400}{3}$ m/sec.
✓ Final answer in radians/sec.

[5]

18 **[3 marks]**

Determine the vector equation of a plane which contains the point A with position vector $\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$ and parallel to the vectors $\begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix}$.

Let $\underset{\sim}{n}$ be the normal vector to the plane, and $\underset{\sim}{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Using dot products: $\underset{\sim}{n} \cdot \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} = 0$ and $\underset{\sim}{n} \begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix} = 0$

$$\therefore \left.\begin{aligned} -a + 5b + 3c &= 0 \\ 2a - 5b &= 0 \end{aligned}\right\} \quad a + 3c = 0$$

$\therefore$ $c = -\frac{1}{3}a$

and $b = \frac{2}{5}a$

$$\therefore \underset{\sim}{n} = \lambda \begin{pmatrix} 15 \\ 6 \\ -5 \end{pmatrix}$$

$$\Rightarrow \underset{\sim}{r} \cdot \begin{pmatrix} 15 \\ 6 \\ -5 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 15 \\ 6 \\ -5 \end{pmatrix}$$

Answer: $\underset{\sim}{r} \cdot \begin{pmatrix} 15 \\ 6 \\ -5 \end{pmatrix} = -8$

OR $\underset{\sim}{r} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -5 \\ 0 \end{pmatrix}$

✓ uses dot products of $\underset{\sim}{n}$ and 2 given vectors.
✓ determines $\underset{\sim}{n}$ appropriately.
✓ determines correct vector equation of plane.

19. [5 marks]

The female population of a species is shown in the table below together with estimates of breeding and survival rates.

Age (years)	0 – 2	2 – 4	4 – 6	6 – 8	8 – 10
Initial Population	1800	1500	1150	700	400
Breeding Rate	0	0.4	1.5	1.2	0.3
Survival Rate	0.6	0.9	0.7	0.5	0

(a) If no harvesting takes place, estimate the long-term population growth rate.

$$L = \begin{bmatrix} 0 & 0.4 & 1.5 & 1.2 & 0.3 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \end{bmatrix} \quad \text{and} \quad P(0) = \begin{bmatrix} 1800 \\ 1500 \\ 1150 \\ 700 \\ 400 \end{bmatrix}$$

✓ sets up L and P(0).
✓ determines 3 consecutive population estimates
✓ Determines long term growth rate as a percentage

$[1\ 1\ 1\ 1\ 1] \times L^{25} \times P(0) = [192050.86]$

$[1\ 1\ 1\ 1\ 1] \times L^{26} \times P(0) = [220965.11] \Rightarrow \dfrac{220965.11}{192050.86} = 1.1506$

$[1\ 1\ 1\ 1\ 1] \times L^{27} \times P(0) = [254243.83] \Rightarrow \dfrac{254243.83}{220965.11} = 1.1506$

$\Rightarrow$ Long term growth rate $= \underline{15.06\%}$ (2dp)

[3]

(b) What percentage of the population will need to be culled in order for the long term population · to be stable?

$P_{n+1} = (1-h) \times 1.1506\, P_n$, where h is the cull rate

However, for stable population, $P_{n+1} = P_n$

$\therefore\ (1-h) \times 1.1506 = 1$

$\Rightarrow\ h = 0.1309$

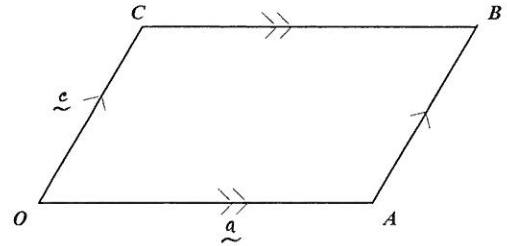
$\Rightarrow$ Cull rate of $\underline{13.09\%}$ (2dp)

[2]

✓ sets up an equation in "h", using $P_{n+1} = P_n$.
✓ determines cull rate for stable population.

20. [5 marks]

Consider the parallelogram shown below. Let $\overrightarrow{OA} = \boldsymbol{a}$ and $\overrightarrow{OC} = \boldsymbol{c}$.



Use vectors to prove that if the diagonals of a parallelogram are perpendicular then the parallelogram is a rhombus.

$\overrightarrow{OB} = \underset{\sim}{a} + \underset{\sim}{c}$ and $\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$

If $\overrightarrow{OB} \perp \overrightarrow{AC}$, then $\overrightarrow{OB} \cdot \overrightarrow{AC} = 0$

ie., $(\underset{\sim}{a} + \underset{\sim}{c}) \cdot (\underset{\sim}{c} - \underset{\sim}{a}) = 0$

$\underset{\sim}{a} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{c} \cdot \underset{\sim}{c} - \underset{\sim}{a} \cdot \underset{\sim}{c} = 0$

$\therefore\ |\underset{\sim}{c}|^2 - |\underset{\sim}{a}|^2 = 0$

$\Rightarrow\ |\underset{\sim}{c}| = |\underset{\sim}{a}|$

$\Rightarrow\ |\overrightarrow{OC}| = |\overrightarrow{OA}|$

$\Rightarrow$ The parallelogram must be a rhombus if the diagonals are perpendicular.

✓ determines vector expressions for diagonals in terms of $\underset{\sim}{a}$ and $\underset{\sim}{c}$.
✓ uses dot product = 0.
✓ deduces that $|\underset{\sim}{c}| = |\underset{\sim}{a}|$ if dot product = 0.
✓ correct conclusion provided.
✓ structure of proof is sound.